PERGAMON

International Journal of Solids and Structures 36 (1999) 5545–5558

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# Mechanical behavior of two kinds of rubber materials

Y.C. Gao[a,*], Tianjie Gao[b]

[a] *Northern Jiaotong University, 100044, Beijing, P.R. China*
[b] *Department of Physics, University of Pennsylvania, PA 19104-6396, U.S.A.*

Received 6 December 1997; in revised form 12 August 1998

**Abstract**

Two kinds of elastic laws have been adopted by Gao (1990 and 1997) to analyze the deformation fields near a crack tip. One of them contains the response to volume change and shape change; the other contains the response to extension and compression. In this paper the two kinds of constitutive relations are examined by typical large deformation, and the restrictions on constitutive parameters are discussed. 

## 1. Introduction

An important problem in nonlinear elastic theory is to give a reasonable and applicable elastic law. Many attempts have been made to develop a theoretical stress–strain relation that can fit experimental results for highly elastic materials. However, in general, the strain energy function is complicated. Ogden (1972a, b) proposed a form of strain energy which is a linear combination of strain invariants. An excellent agreement between Ogden's formula and Treloar's (1958) experimental data up to 7 times extension is obtained.

When we consider a problem with singular point (such as crack tip, concentrated force), the situation is different from the ordinary finite deformation case. Actually, near a singular point in rubber like material, the strain has a tendency to go to infinity, that renders the problem complicated. Theoretical analysis requires the elastic law to be expressed as simple as possible. However, the simplicity often violates rationality. In order to reflect the material behavior near a singular point, i.e. strain tends to infinity, two kinds of constitutive relations were introduced by Gao (1990 and 1997); the earlier one represents the response to shape change and to volume change; the latter represents the response to tension and compression. These two constitutive relations were successfully used in the analysis of a singular point by Gao and Gao (1994), Gao and Shi (1995), Gao and Liu (1995, 1996), Gao and Gao (1996), Gao (1997). Furthermore, these two kinds of

* Corresponding author. Tel.: 00 86 10 63240427; fax: 00 86 10 62255671; e-mail: ycgao@center.njtu.edu.cn


PII: S0020-7683(98)00259-5

strain energy were directly used in finite element calculation and were shown to be stable by Zhou and Gao (1998) in the Total Lagrangian (L.T.) method, the maximum tension of a line element reached 50 times. It is well known that every constitutive relation contains some parameters that are restrained in certain value reach to ensure the material behavior to be reasonable. In the present paper the two elastic laws will be examined by typical load. The constitutive parameters will be discussed in detail. Finally, these two elastic laws are compared with Ogden's (1972a, b) formulae and Knowles et al. (1973) formulae.

## 2. Basic formulae

Let $P$ and $Q$ denote the position vectors of a material point before and after deformation, respectively, $x^i$ $(i = 1, 2, 3)$ is the Lagrangian coordinate. Two sets of local triads are defined as follows,

$$P_i = \frac{\partial P}{\partial x^i}, \quad Q_i = \frac{\partial Q}{\partial x^i} \tag{1}$$

The displacement gradient tensor is

$$F = Q_i \otimes P^i \tag{2}$$

Where $P^i$ is the conjugate of $P_i$, $\otimes$ the dyadic symbol, and the summation rule is implied. The right and left Cauchy–Green strain tensors are

$$D = F^{\mathrm{T}} \cdot F, \quad d = F \cdot F^{\mathrm{T}} \tag{3}$$

where superscript T indicates transposition. $D$ and $d$ possess the same invariants such as

$$I_1 = D:U = d:U, \quad I_2 = D:D = d:d, \quad I_3 = D^2:D = d^2:d, \ldots \tag{4}$$

$$I_{-1} = D^{-1}:U = d^{-1}:U, \quad I_{-2} = D^{-1}:D^{-1} = d^{-1}:d^{-1}, \ldots \tag{5}$$

in which $U$ denotes unit tensor, : denotes duel multiplication. Among these invariants, there are only three independent. Besides, a common used invariant is $K$,

$$K = \tfrac{1}{6}(I_1^3 - 3I_1 I_2 + 2I_3) \tag{6}$$

Let $\lambda_i$ denote the values of principal strain, then

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_{-1} = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}, \quad K = \lambda_1^2 \lambda_2^2 \lambda_3^2 \tag{7}$$

For isotropic material, the strain energy per unit undeformed volume can be expressed by three independent invariants, for example,

$$W = W(I_1, I_2, K) \tag{8}$$

The Kirchhoff stress is

$$\sigma = 2\frac{\partial W}{\partial D} \tag{9}$$

then the Cauchy stress is

$$\tau = K^{-1/2} F \cdot \sigma \cdot F^{\mathrm{T}} \tag{10}$$

## 3. Two elastic laws

From physics point of view, the necessary conditions for a reasonable constitutive relation of a solid can be stated in two ways:

(1) A material element must possess stiffness to resist both shape change and volume change.
(2) A material element must possess stiffness to resist both extension and compression.

According to statement (1), a strain energy formula that only contains two terms was proposed by Gao (1990),

$$W = a\left(\frac{I_1}{K^{1/3}}\right)^n + b(K-1)^m K^{-q} \tag{11}$$

where $a$, $b$, $n$, $m$, $q$ are positive constitutive parameters $m$ should be an even integer. Noting eqn (7), it is found that the first term of eqn (11) only depends on the ratio of $\lambda_i$ but not on their magnitudes, therefore, it only reflects pure shape change; the second term of eqn (11) reflects pure volume change. Equations (9)–(11) can be used to give

$$\tau = 2K^{-1/2}\left\{na\left(\frac{I_1}{K^{1/3}}\right)^n\left(\frac{d}{I_1}-\frac{U}{3}\right)+b(K-1)^m K^{1-q}\left(\frac{m}{K-1}-\frac{q}{K}\right)U\right\} \tag{12}$$

Evidently, the first term in the brackets is a stress deviator while the second term is a hydrostatic stress.

According to statement (2), another strain energy formula that is even simpler was proposed by Gao (1997),

$$W = A(I_1^N + I_{-1}^N) \tag{13}$$

Using equations (9), (10) and (13) we obtain

$$\tau = 2NAK^{-1/2}(I_1^{N-1}d - I_{-1}^{N-1}d^{-1}) \tag{14}$$

## 4. Material behavior according to eqn (12)

### 4.1. Small strain case

Let

$$\varepsilon = \tfrac{1}{2}(d-U), \quad e = \varepsilon : U \tag{15}$$

Assuming $\varepsilon$ to be small, eqn (12) is reduced to

$$\tau = 4na3^{n-1}\left(\varepsilon - \frac{e}{3}U\right) + 2^m bme^{m-1}U \tag{16}$$

For $m = 2$, the linear elastic relation is obtained, and it follows that

$$v = \frac{2b - n3^{n-2}a}{4b + n3^{n-2}a}, \quad E = \frac{8n3^n ab}{4b + n3^{n-2}a} \tag{17}$$

where $E$ and $v$ are Young's modulus and Poisson's ratio, respectively. Evidently, $v < 0.5$ is automatically satisfied. The condition $v > 0$ requires that

$$b > \frac{n}{2}3^{n-2}a \tag{18}$$

### 4.2. Uniaxial stress

Without loss of generality, we consider a cubic material element with arrises of unit length as shown in Fig. 1(a). Under the action of normal stress $\tau^{11}$ along $x^1$-direction, the arris lengths become $\lambda$, $\mu$ and $\mu$, respectively, but the arrises still remain perpendicular, as shown in Fig. 1(b). Let $e_i$ $(i = 1, 2, 3)$ denote the unit vectors along $x^i$-direction, then according to eqns (2)–(7), it follows that

$$F = \lambda e_1 \otimes e_1 + \mu(e_2 \otimes e_2 + e_3 \otimes e_3) \tag{19}$$

$$d = \lambda^2 e_1 \otimes e_1 + \mu^2(e_2 \otimes e_2 + e_3 \otimes e_3) \tag{20}$$

$$d^{-1} = \lambda^{-2} e_1 \otimes e_1 + \mu^{-2}(e_2 \otimes e_2 + e_3 \otimes e_3) \tag{21}$$

$$I_1 = \lambda^2 + 2\mu^2, \quad I_{-1} = \lambda^{-2} + 2\mu^{-2}, \quad K = \lambda^2\mu^4 \tag{22}$$

Further, eqn (12) becomes

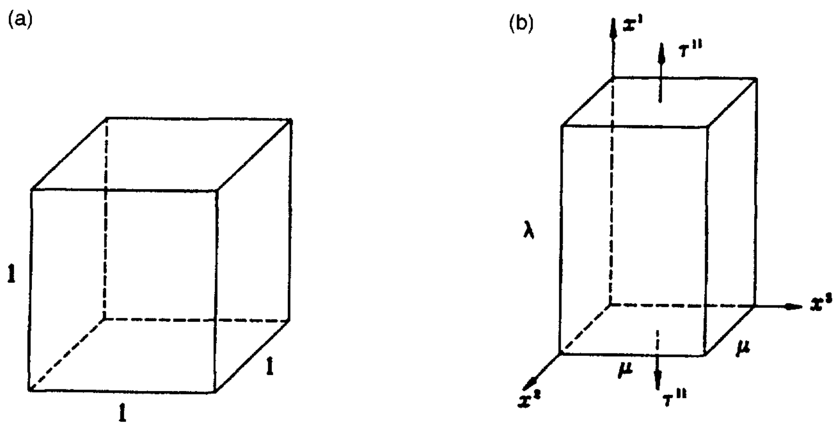


Fig. 1. (a) A material cubic element. (b) Uniaxial load.

$$\tau = \frac{2}{\lambda\mu^2}\left\{\frac{na(\lambda^2-\mu^2)}{3(\lambda^2+2\mu^2)}\left[\left(\frac{\lambda}{\mu}\right)^{4/3}+2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n(2e_1\otimes e_1-e_2\otimes e_2-e_3\otimes e_3)\right.$$
$$\left.+b[(m-q)\lambda^2\mu^4+q](\lambda^2\mu^4-1)^{m-1}(\lambda\mu^2)^{-2q}U\right\} \quad (23)$$

Considering $\tau^{22}=\tau^{33}=0$, eqn (23) gives

$$\frac{na(\lambda^2-\mu^2)}{3(\lambda^2+2\mu^2)}\left[\left(\frac{\lambda}{\mu}\right)^{4/3}+2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n-b[(m-q)\lambda^2\mu^4+q](\lambda^2\mu^4-1)^{m-1}(\lambda\mu^2)^{-2q}=0 \quad (24)$$

then eqn (23) becomes

$$\tau = \frac{2na}{\lambda\mu^2}\frac{(\lambda^2-\mu^2)}{\lambda^2+2\mu^2}\left[\left(\frac{\lambda}{\mu}\right)^{4/3}+2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n e_1\otimes e_1 \quad (25)$$

When $\lambda \gg 1(\mu \ll 1)$, eqn (24) can be used to give

$$\mu = \left(\frac{na}{3sb}\right)^{\frac{3}{4(3s+n)}}\lambda^{-\frac{3s-2n}{2(3s+n)}} \quad (26)$$

where

$$s = m-q \quad (27)$$

eqn (26) shows that when $\lambda \to \infty$ if $\mu \to 0$, the following condition is required

$$s > 2n/3 \quad (28)$$

Substituting eqn (26) into (25), it follows that,

$$\tau = 2(na)^{\alpha}(3sb)^{1-\alpha}\lambda^{2n\alpha}e_1\otimes e_1 \quad (29)$$

in which

$$\alpha = \frac{3}{2}\cdot\frac{2s-1}{3s+n} \quad (30)$$

In order to ensure that $\tau \to \infty$ when $\lambda \to \infty$, $s > 0.5$ is required.

The total load acting on the element is

$$L = \mu^2\tau^{11} = 2(na)^{\frac{3s}{3s+n}}(3sb)^{\frac{n}{3s+n}}\lambda^{\frac{3s(2n-1)-n}{3s+n}} \quad (31)$$

eqn (31) shows that if $L$ increase with $\lambda$, the following condition is required,

$$s > \frac{n}{3(2n-1)} \quad (32)$$

The work to extend the specimen is

$$W = \int_1^{\lambda_*} L\,\mathrm{d}\lambda \doteq \frac{3s+n}{3sn}(na)^{\frac{3s}{3s+n}}(3sb)^{\frac{n}{3s+n}}\lambda_*^{\frac{6sn}{3s+n}} \tag{33}$$

Evidently, when $\lambda_* \to \infty$, $W \to \infty$.

When $\lambda \ll 1(\mu \gg 1)$, eqn (24) gives

$$\mu = \left(\frac{6qb}{2^n na}\right)^{\frac{3}{2(6q+n)}} \lambda^{-\frac{3q-n}{6q+n}} \tag{34}$$

eqn (34) shows that when $\lambda \ll 1$ if $\mu \gg 1$ the following is required,

$$q > n/3 \tag{35}$$

Substituting eqn (34) into (25) it follows that

$$\tau = -(2^n na)^{\beta}(6qb)^{1-\beta}\lambda^{-n\beta} e_1 \otimes e_1 \tag{36}$$

where

$$\beta = \frac{3(2q+1)}{6q+n} \tag{37}$$

The total load acting on the element is

$$L = \mu^2 \tau^{11} = -(2^n na)^{\frac{6q}{6q+n}}(6qb)^{\frac{n}{6q+n}}\lambda^{-\frac{6q(n+1)+n}{6q+n}} \tag{38}$$

eqn (38) shows that the material element is always stable since $n$, $q > 0$.

The work to compress a specimen to become a plate with thickness $\lambda_*$ is

$$W = \int_1^{\lambda_*} L\,\mathrm{d}\lambda \doteq \frac{6q+n}{6qn}(2^n na)^{\frac{6q}{6q+n}}(6qb)^{\frac{n}{6q+n}}\lambda_*^{\frac{-6qn}{6q+n}} \tag{39}$$

### 4.3. Biaxial stress

Consider the same cubic material element as shown in Fig. 1(a). Under the action of normal stress $\tau^{22}$ and $\tau^{23}$ $(=\tau^{22})$, the arris length becomes $\lambda$, $\mu$ and $\mu$, respectively, as shown in Fig. 2. Equations (19)–(23) are still valid but the relation of $\lambda$ and $\mu$ must be given by $\tau^{11} = 0$, i.e.

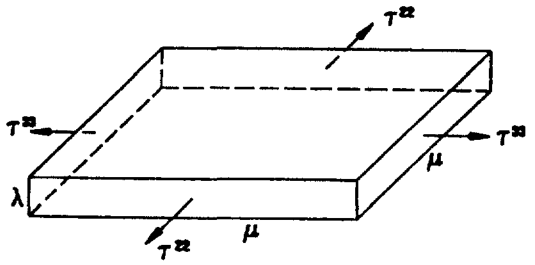


Fig. 2. Biaxial load.

$$\frac{2na(\lambda^2-\mu^2)}{3(\lambda^2+2\mu^2)}\left[\left(\frac{\lambda}{\mu}\right)^{4/3}+2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n+b[(m-q)\lambda^2\mu^4+q](\lambda^2\mu^4-1)^{m-1}(\lambda\mu^2)^{-2q}=0 \tag{40}$$

then eqn (23) is reduced to

$$\tau=\frac{2na}{\lambda\mu^2}\frac{(\mu^2-\lambda^2)}{\lambda^2+2\mu^2}\left[\left(\frac{\lambda}{\mu}\right)^{4/3}+2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n(e_2\otimes e_2+e_3\otimes e_3) \tag{41}$$

When $\mu\gg 1(\lambda\ll 1)$, eqn (40) gives

$$\lambda=\left(\frac{2^n na}{3sb}\right)^{\frac{3}{2(3s+n)}}\mu^{-\frac{6s-n}{3s+n}} \tag{42}$$

then eqn (41) becomes

$$\tau=(2^n na)^{\alpha}(3sb)^{1-\alpha}\mu^{2n\alpha}(e_2\otimes e_2+e_3\otimes e_3) \tag{43}$$

where the $\alpha$ is given by eqn (30).

The total load on one side is

$$L=\lambda\mu\tau^{22}=\lambda\mu\tau^{33}=(2^n na)^{\frac{3s}{3s+n}}(3sb)^{\frac{n}{3s+n}}\mu^{\frac{3s(2n-1)-n}{3s+n}} \tag{44}$$

eqn (44) shows that if $L$ increase with $\mu$, the condition (32) is required.

The work to extend the specimen is

$$W=2\int_1^{\mu_*}L\,\mathrm{d}\mu\doteq\frac{3s+n}{3sn}(2^n na)^{\frac{3s}{3s+n}}(3sb)^{\frac{n}{3s+n}}\mu_*^{\frac{6sn}{3s+n}} \tag{45}$$

When $\mu\ll 1(\lambda\gg 1)$, eqn (40) gives

$$\lambda=\left(\frac{3qb}{2na}\right)^{\frac{3}{2(3q+2n)}}\mu^{-\frac{2(3q-n)}{3q+2n}} \tag{46}$$

eqn (46) shows that if eqn (35) is satisfied, $\lambda\to\infty$ when $\mu\to 0$. Substituting eqn (46) into (41) it follows that

$$\tau=-(2na)^{\gamma}(3qb)^{1-\gamma}\mu^{-4n\gamma}(e_2\otimes e_2+e_3\otimes e_3) \tag{47}$$

where

$$\gamma=\frac{3(2q+1)}{2(3q+2n)} \tag{48}$$

The load on one side is

$$L=\lambda\mu\tau^{22}=-(2na)^{\frac{3q}{3q+2n}}(3qb)^{\frac{2n}{3q+2n}}\mu^{-\frac{3q(4n+1)+2n}{3q+2n}} \tag{49}$$

The work to compress the specimen is

$$W = 2\int_1^{\mu_*} L\,\mathrm{d}\mu \doteq \frac{3q+2n}{6qn}(2na)^{\frac{3q}{3q+2n}}(3qb)^{\frac{2n}{3q+2n}}\mu_*^{-\frac{12qn}{3q+2n}} \tag{50}$$

## 5. Material behavior according to eqn (14)

### 5.1. Small strain case

From eqn (15) it follows that

$$d^{-1} = U - 2\varepsilon, \quad I_{-1} = 3 - 2e \tag{51}$$

Substituting eqns (15) and (51) into (14) it follows that

$$\tau = 8NA3^{N-1}\left[\varepsilon + \frac{N-1}{3}eU\right] \tag{52}$$

then

$$E = \frac{8N^2}{2N+1}3^N A, \quad \nu = \frac{N-1}{2N+1} \tag{53}$$

The condition $\nu > 0$ requires

$$N > 1 \tag{54}$$

### 5.2. Uniaxial stress

Consider the specimen shown in Fig. 1. Equations (19)–(23) are still valid, then eqn (14) gives

$$\tau = \frac{2NA}{\lambda\mu^2}[(I_1^{N-1}\lambda^2 - I_{-1}^{N-1}\lambda^{-2})e_1 \otimes e_1 + (I_1^{N-1}\mu^2 - I_1^{N-1}\mu^{-2})(e_2 \otimes e_2 + e_3 \otimes e_3)] \tag{55}$$

The conditions $\tau^{22} = \tau^{33} = 0$ give

$$(\lambda^2 + 2\mu^2)^{N-1}\mu^2 - (\lambda^{-2} + 2\mu^{-2})^{N-1}\mu^{-2} = 0 \tag{56}$$

then eqn (55) becomes

$$\tau = \frac{2NA}{\lambda\mu^2} I_1^{N-1}\left(\lambda^2 - \frac{\mu^4}{\lambda^2}\right)e_1 \otimes e_1 \tag{57}$$

When $\lambda \gg 1 (\mu \ll 1)$, eqn (56) gives

$$\mu = 2^{\frac{N-1}{2(N+1)}}\lambda^{-1+\frac{2}{N+1}} \tag{58}$$

If $\mu \to 0$ when $\lambda \to \infty$, the condition (54) is required. From eqns (22) and (58) we can see, if $K$ increase with $\lambda$ the following condition is required,

$$N < 3 \tag{59}$$

Equations (57) and (58) can be combined to give

$$\tau = 2^{\frac{2}{N+1}} AN\lambda^{2N+1-\frac{4}{N+1}} e_1 \otimes e_1 \tag{60}$$

The total load is

$$L = \mu^2 \tau^{11} = 2NA\lambda^{2N-1} \tag{61}$$

The work to extend the specimen is

$$W = \int_1^{\lambda_*} L\,d\lambda \doteq A\lambda_*^{2N} \tag{62}$$

When $\lambda \ll 1 (\mu \gg 1)$, eqn (56) gives

$$\mu = 2^{-\frac{N-1}{2(N+1)}} \lambda^{-1+\frac{2}{N+1}} \tag{63}$$

then eqn (57) is reduced to

$$\tau = -2^{\frac{2N}{N+1}} AN\lambda^{-2N+1-\frac{4}{N+1}} e_1 \otimes e_1 \tag{64}$$

The total load is

$$L = \mu^2 \tau^{11} = -2NA\lambda^{-2N-1} \tag{65}$$

The work to compress the specimen is

$$W = \int_1^{\lambda_*} L\,d\lambda \doteq A\lambda_*^{-2N} \tag{66}$$

### 5.3. *Biaxial stress*

Consider the cubic material element as shown in Fig. 1(a), under the action of $\tau^{22}$ and $\tau^{33}$ $(=\tau^{22})$ it is deformed as shown in Fig. 2. Using the condition $\tau^{11} = 0$ and eqn (55) we obtain

$$(\lambda^2 + 2\mu^2)^{N-1}\lambda^2 - (\lambda^{-2} + 2\mu^{-2})^{N-1}\lambda^{-2} = 0 \tag{67}$$

then eqn (55) is reduced to

$$\tau = \frac{2NA}{\lambda\mu^2} I_1^{N-1} \left(\mu^2 - \frac{\lambda^4}{\mu^2}\right)(e_2 \otimes e_2 + e_3 \otimes e_3) \tag{68}$$

When $\mu \gg 1 (\lambda \ll 1)$, eqn (67) gives

$$\lambda = 2^{-\frac{N-1}{2(N+1)}} \mu^{-1+\frac{2}{N+1}} \tag{69}$$

Then eqn (68) is reduced to

$$\tau = 2^{N+\frac{N-1}{2(N+1)}} A N \mu^{2N-1-\frac{2}{N+1}} (e_2 \otimes e_2 + e_3 \otimes e_3) \tag{70}$$

The load on the side is

$$L = \lambda \mu \tau^{22} = N A 2^N \mu^{2N-1} \tag{71}$$

The work to extend the specimen is

$$W = 2 \int_1^{\mu_*} L \, \mathrm{d}\mu \doteq 2^N A \mu_*^{2N} \tag{72}$$

When $\mu \ll 1 (\lambda \gg 1)$, eqn (67) gives

$$\lambda = 2^{\frac{N-1}{2(N-1)}} \mu^{-1+\frac{2}{N+1}} \tag{73}$$

then (68) is reduced to

$$\tau = -2^{N-\frac{N-1}{2(N+1)}} A N \mu^{-2N-1-\frac{2}{N+1}} (e_2 \otimes e_2 + e_3 \otimes e_3) \tag{74}$$

The load on the side is

$$L = \lambda \mu \tau^{22} = -2^N N A \mu^{-2N-1} \tag{75}$$

The work to compress the specimen is

$$W = 2 \int_1^{\mu_*} L \, \mathrm{d}\mu \doteq 2^N A \mu_*^{-2N} \tag{76}$$

## 6. An example

We consider a spherical rubber membrane subject to internal pressure $p$ as shown in Fig. 3. Let $(R, \Theta, \Phi)$ and $(r, \theta, \varphi)$ denote the spherical coordinates for the framework before and after deformation, respectively. The deformation is described by the mapping function

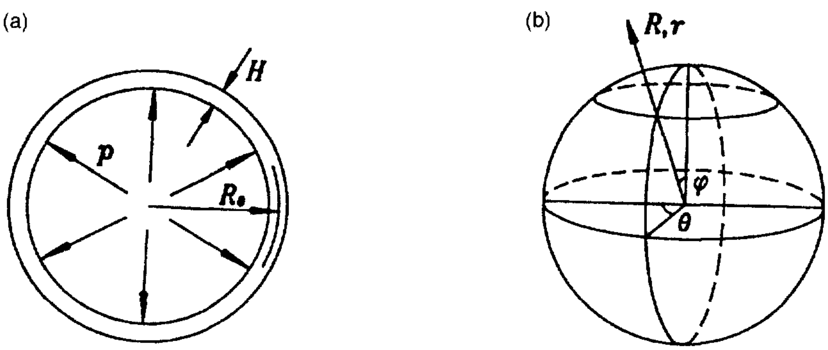


Fig. 3. (a) A thin spherical membrane. (b) Spherical coordinates.

$$\begin{cases}\theta = \Theta \\ r = f(R)\end{cases}, \quad \varphi = \Phi \tag{77}$$

Let $e_i$ $(i = r, \theta, \varphi)$ denote the unit vectors along the coordinate lines $r$, $\theta$ and $\varphi$, then

$$P_R = e_r, \quad P_\Theta = R\sin\Phi e_\theta, \quad P_\Phi = Re_\varphi \tag{78}$$

$$Q_R = \frac{\mathrm{d}f}{\mathrm{d}R}e_r = f'e_r, \quad Q_\Theta = r\sin\Phi e_\theta, \quad Q_\Phi = re_\varphi \tag{79}$$

therefore,

$$d = \lambda^2 e_r \otimes e_r + \mu^2(e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) \tag{80}$$

where

$$\lambda = f', \quad \mu = \frac{f}{R} \tag{81}$$

Then $I_1$, $I_{-1}$ and $K$ are given by eqn (22). It is assumed that initial thickness $H$ is much smaller than the radius of the membrane $R_0$, so that the strain and stress along the thickness can be considered as constants, i.e. the values of $\lambda$ and $\mu$ in eqn (81) are constants.

Further, since $H \ll R_0$, we assume that

$$\tau^{rr} \ll \tau^{\theta\theta}(=\tau^{\varphi\varphi}) \tag{82}$$

then the equilibrium condition for the membrane can be written as

$$\tau^{\theta\theta} + \tau^{\varphi\varphi} - \frac{r}{h}p = 0 \tag{83}$$

where $h$ is the thickness after deformation,

$$h = Hf' = \lambda H \tag{84}$$

The analysis of deformation will be given according to two elastic laws of eqn (12) and eqn (14), respectively.

### 6.1. According to eqn (12)

From eqns (41), (83), (84), and (81) it follows that

$$\frac{2na}{\lambda\mu^2}\frac{\mu^2-\lambda^2}{\lambda^2+2\mu^2}\left[\left(\frac{\lambda}{\mu}\right)^{4/3} + 2\left(\frac{\mu}{\lambda}\right)^{2/3}\right]^n = \frac{\mu Rp}{2\lambda H} \tag{85}$$

When $\mu \gg 1$, eqn (41) becomes eqn (43) then eqn (85) is reduced to

$$\frac{Rp}{2H} = (2^n na)^{\frac{3s}{3s+n}}(3sb)^{\frac{n}{3s+n}}\mu^{\frac{3}{3s+n}[s(2n-3)-n]} \tag{86}$$

If $p$ increase with $\mu$, the following condition is required

$$n > 3/2, \quad s > \frac{n}{2n-3} \tag{87}$$

The work to swell up the membrane is

$$W = 4\pi R^3 \int_1^{\mu_*} \mu^2 p \, d\mu \doteq 4\pi R^2 H \frac{3s+n}{3sn} (2^n na)^{\frac{3s}{3s+n}} (3sb)^{\frac{n}{3s+n}} \mu_*^{\frac{6sn}{3s+n}} \tag{88}$$

*6.2. According to eqn (14)*

From eqns (57), (83), (84) and (81) it follows that

$$\frac{2NA}{\lambda\mu^2} I_1^{N-1} \left(\mu^2 - \frac{\lambda^4}{\mu^2}\right) = \frac{\mu Rp}{2\lambda H} \tag{89}$$

When $\mu \gg 1$, eqn (89) is reduced to

$$\frac{Rp}{2H} = 2^N NA\mu^{2N-3} \tag{90}$$

If $p$ increase with $\mu$ it is required that

$$N > 3/2 \tag{91}$$

The work to swell up the membrane is

$$W = 4\pi R^3 \int_1^{\mu_*} \mu^2 p \, d\mu \doteq 4\pi 2^N R^2 HA\mu_*^{2N} \tag{92}$$

## 7. The relation of eqns (11) and (13) with other forms of energy

Ogden (1972a, b) proposed a quite general but convenient form of strain energy, using the notation of this paper, it can be written as

$$W = \sum_i \mu_i \phi(\alpha_i) + F(K) \tag{93}$$

where $K$ is given in (7), $\mu_i$ are constants, $\phi(\alpha_i)$ are strain invariants with exponent $\alpha_i$ that may not be integer,

$$\phi(\alpha_i) = (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3)/\alpha_i \tag{94}$$

If $F \equiv 0$, and only two terms in (93) are taken, let,

$$\alpha_1 = 2, \quad \alpha_2 = -2, \quad \mu_1 = 2A, \quad \mu_2 = -2A \tag{95}$$

then (93) becomes an equivalent form of (13) for the case of $N = 1$. Ogden (1972a, b) obtained a sufficient condition for satisfying Hill's constitutive inequality,

$$\mu_i\alpha_i > 0 \quad \text{(each } i\text{, no summation)} \tag{96}$$

Evidently, eqn (13) meets this condition for $N = 1$.

As for the general case, $1 < N < 3$, eqn (13) is not equivalent with (93), the Hill's inequality is difficult to discuss, but the analysis in this paper directly revealed the reasonable reach of $N$.

From Ogden's original idea, see eqn (4) of Ogden (1972b), the general form of (93) can be written as

$$W = \sum_i \mu_i \phi(\alpha_i) \cdot K^{\gamma_i} + F(K) \tag{97}$$

If only one $\phi(\alpha_i)$ is taken and $\alpha_1 = 2$, $\gamma_1 = -1/3$, $\mu_1 = 2a$, let

$$F = b(K-1)^m K^{-q} + 3aK^{-1/3} \tag{98}$$

then (97) becomes a special case of (11), i.e. $n = 1$.

Therefore, both eqns (11) and (13) possess some common feature with the energy form given by Ogden (1972a, b).

Knowles and Sternberg (1973) proposed another form of energy and has been used by many authors,

$$W = (AI_1 + BJ + CI_1 J^{-2})^N \tag{99}$$

where

$$J = K^{1/2} \tag{100}$$

The Hill's inequality was not discussed by Knowles and Strenberg (1973) for (99). The interesting fact is that the same essential solutions were obtained according to different energy forms (11), (13) and (99), see Gao and Gao (1999).

The inherent relations between (11), (13), (97) and (99) should be further discussed in concrete application.

## Acknowledgement

This work is supported by the National Science Foundation of China (No. 19572001).

## References

Gao, Y.C., 1990. Elastostatic crack tip behavior for a rubber like material. Theoretical and Appl. Fract. Mech. 14, 219–231.

Gao, Y.C., 1997. Large deformation field near a crack tip in rubber-like material. Theoretical and Appl. Fract. Mech. 26, 155–162.

Gao, Y.C., Gao, T.J., 1999. Analytical solution to a notch tip field in rubber-like materials under tension. Int. J. Solids Structures 36, 5559–5571.

Gao, T.S., Gao, Y.C., 1994. A rubber wedge under compression of a line load. Int. J. Solids Structures 31 (17) 2393–2406.

Gao, Y.C., Gao, T.S., 1996. Notch-tip field in rubber-like materials under tension and shear mixed load. Int. J. Fracture 78, 283–298.

Gao, Y.C., Liu, B., 1996. Stress singularity near the notch tip of a rubber like specimen under tension. Eur. J. Mech., A/Solids 15 (2), 199–211.

Gao, Y.C., Liu, B., 1995. A rubber cone under the tension of a concentrated force. Int. J. Solids Structures 32 (11), 1485–1493.

Gao, Y.C., Shi, Z.F., 1995. Large strain field near an interface crack tip. Int. J. Fracture 69, 269–279.

Knowles, J.K., Sternberg, E., 1973. An asymptotic finite deformation analysis of the elastostatic field near the tip of a crack. J. of Elasticity 3, 67–107.

Ogden, R.W., 1972a. Large deformation isotropic elasticity—on the correlation of theory and experiment for incompressible rubber-like solids. Proc. R. Soc. Lond. A. 326, 565–584.

Ogden, R.W., 1972b. Large deformation isotropic elasticity—on the correlation of theory and experiment for compressible rubber-like solids. Proc. R. Soc. Lond. A. 328, 567–583.

Treloar, L.R.G., 1958. The Physics of Rubber Elasticity. 2nd ed. Oxford University Press.

Zhou, Z., Gao, Y.C., 1998. Analysis and finite element calculation of crack tip field in rubber like material under tension and shear mixed load, to be published.